Complementary Tree Nil Domination Number of Circular-Arc Graphs

S. Muthammai¹, G. Ananthavalli²
¹²Government Arts College for Women (Autonomous), Pudukkottai, India
¹muthammai.sivakami@gmail.com, ²dv.ananthavalli@gmail.com

Abstract: A set \( D \subseteq V \) of a graph \( G = (V, E) \) is a dominating set, if every vertex in \( V(G) - D \) is adjacent to some vertex in \( D \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set. A dominating set \( D \) of a connected graph \( G \) is called a complementary tree nil dominating set if the induced subgraph \( <V(G) - D> \) is a tree and \( V(G) - D \) is not a dominating set. The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of \( G \) and is denoted by \( \gamma_{ctnd}(G) \). In this paper, some results regarding the complementary tree nil domination number of circular-arc graphs are found.

Keywords: Complementary tree domination number, complementary tree nil domination number, Circular-arc Graph.

I. INTRODUCTION

Graphs discussed in this paper are finite, undirected and simple connected graphs. For a graph \( G \), let \( V(G) \) and \( E(G) \) denote its vertex set and edge set respectively. A graph with \( p \) vertices and \( q \) edges is denoted by \( G(p, q) \).

For \( v \in V(G) \), the neighborhood \( N(v) \) of \( v \) is the set of all vertices adjacent to \( v \) in \( G \). \( N[v] = N(v) \cup \{v\} \) is called the closed neighborhood of \( v \). For any two vertices \( u \), \( v \) in \( G \), if there exists at least one \( u - v \) path, the distance \( d(u, v) \) between \( u \) and \( v \) is the minimum length of a \( u - v \) path. The eccentricity of a vertex \( v \) of a connected graph \( G \) is \( e(v) = \max \{d(u, v); v \in E(V(G))\} \). The radius of \( G \) is \( \text{rad}(G) = \min \{e(v); v \in V(G)\} \) and the diameter of \( G \) is \( \text{diam}(G) = \max \{e(v); v \in V(G)\} \). A vertex \( v \in V(G) \) is called a support if it is adjacent to a pendant vertex. That is, a vertex of degree one.

The concept of domination in graphs was introduced by Ore[3]. A set \( D \subseteq V \) of \( G \) is said to be a dominating set of \( G \) if every vertex in \( V(G) - D \) is adjacent to some vertex in \( D \). The cardinality of a minimum dominating set in \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \).

Some domination parameters are defined by imposing additional constraints on the complement of a dominating set. Such parameters are called codomination parameters. Based on these, the concept of nonsplit domination in graphs was introduced by Kulli and Janakiram [4]. A dominating set \( D \) of a connected graph \( G \) is a nonsplit dominating set, if the induced subgraph \( <V(G) - D> \) is connected. Complementary nil domination number of a graph was defined and studied by T. Tamil Chelvam and S. Robinson Chellathurai [5]. A set \( D \subseteq V \) of \( G \) is said to be a complementary nil dominating set (ctnd-set) of a graph \( G \) if it is a dominating set and its complement \( V - D \) is not a dominating set for \( G \). The minimum cardinality of a ctnd-set is called the complementary nil domination number of \( G \) and is denoted by \( \gamma_{ctnd}(G) \).

Muthammai, Bhunumathi and Vidhya[5] introduced the concept of complement tree dominating set. A dominating set \( D \subseteq V \) of \( G \) is said to be a complement tree dominating set (ctd-set) if the induced subgraph \( <V(G) - D> \) is a tree. The minimum cardinality of a ctd-set is called the tree dominating number of \( G \) and is denoted by \( \gamma_{ctd}(G) \). We introduced the concept of complementary tree nil dominating set [7]. A dominating set \( D \subseteq V \) of \( G \) is said to be a complementary tree nil dominating set (ctnd-set) if the induced subgraph \( <V(G) - D> \) is a tree and \( V(G) - D \) is not a dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of \( G \) and is denoted by \( \gamma_{ctnd}(G) \).

Circular-arc graphs are a new class of intersection graphs, defined for a set of arcs on a circle. A graph is a circular-arc graph, if it is the intersection graph of a finite set of arcs on a circle. That is, there exists one arc for each vertex of \( G \) and two vertices in \( G \) are adjacent in \( G \), if and only if the corresponding arcs intersect. Let \( A = \{A_1, A_2, ..., A_n\} \) be a circular-arc family on a circle, where all the arcs together cover the entire circle. An arc \( A_i \) and \( A_j \) are said to intersect each other if they have nonempty intersection.

II. PRIOR RESULTS

Theorem 2.1. [5] Let \( G \) be a connected graph with \( p \geq 4 \). Then \( \gamma_{ctd}(G) = p - 1 \) if and only if \( G \) is a star on \( p \) vertices.

Theorem 2.2. [7] A complementary tree nil dominating set \( D \) of a connected graph \( G \) is minimal if and only if for each vertex \( v \) in \( D \), one of the following conditions holds.

(a) \( v \) is an isolated vertex of \( D \).

(b) There exists a vertex \( u \) in \( V - D \) such that \( N(u) \cap D = \{v\} \).

(c) \( V - (D - \{v\}) \) is a dominating set of \( G \).

(d) \( V - (D - \{v\}) \) either contains cycle or disconnected.

Theorem 2.3. [7] For any connected graph \( G \) with \( p \) vertices, \( 2 \leq \gamma_{ctnd}(G) \leq p \), where \( p \geq 2 \).
Theorem 2.4. [7] Let $D$ be a ctn of a connected graph $G$ and $S$ be the set of all pendant vertices in $<V(G) - D>$. If there exists a vertex $v \in D$ such that $N(v) \cap (V(G) - D) \subseteq S$, then $\gamma_{cnd}(G) \leq \gamma_{ctn}(G) + m$, where $m = |N(v) \cap (V - D)|$.

Theorem 2.5. [7] For any connected graph $G$ with $\delta(G) \geq 2$, $\beta_D(G) + 1 \leq \gamma_{ctn}(G)$.

Theorem 2.6. [7] If $G$ is a connected graph with $\gamma_{ctn}(G) = 3$ and if $\delta(G) \geq 2$, then $\text{diam}(G) \leq 4$.

Theorem 2.7. [7] Let $D$ be a ctn of a connected graph $G$ with $\rho$ being the family of all pendant vertices in $G$. Therefore $\gamma_{ctn}(G) = 3$ and if $\delta(G) \geq 2$, then $\text{diam}(G) \leq 4$.

III. MAIN RESULTS

In the following, a necessary and sufficient condition that $\gamma_{cnd}(G) = n$ for a circular-arc graph is obtained.

Theorem 3.1: Let $A = \{A_1, A_2, ..., A_n\}$, $(n \geq 3)$ be a circular-arc family corresponding to a circular-arc graph $G$. Then $\gamma_{cnd}(G) = n$ if and only if each arc $A_i \in A$ dominates all other arcs in $A$.

Proof: Let $A = \{A_1, A_2, ..., A_n\}$, $(n \geq 3)$ be a circular-arc family corresponding to a circular-arc graph $G$.

Assume that $\gamma_{cnd}(G) = n$. Let $A_1, A_2 \in A$ be two nonintersecting arcs in $A$. Consider the graph $G - \{A_1, A_2\}$. Then $\gamma_{cnd}(G) \leq n-1$, which is a contradiction. Hence each arc $A_i \in A$ dominates all other arcs in $A - \{A_i\}$.

Conversely, assume each arc $A_i \in A$ dominates all other arcs in $A$. Then $\gamma_{cnd}(G) = n$.

Example 3.1: Consider the circular-arc family $A = \{A_1, A_2, A_3, A_4\}$ corresponding to a circular-arc graph $G$ given in Fig (3.1).

Fig (3.1)

This circular-arc family $A$ satisfies all the conditions mentioned in Theorem 3.1, for $n = 5$. The complementary tree nil domination number of the circular-arc graph $G$ corresponding to this family is 5.

In the following, a necessary and sufficient condition that $\gamma_{cnd}(G) = n-1$, $(n \geq 4)$ for a circular-arc graph $G$ is obtained.

Theorem 3.2: Let $A = \{A_1, A_2, ..., A_n\}$, $(n \geq 4)$ be a circular-arc family corresponding to a circular-arc graph $G$, $G \notin K_n$, $(n \geq 4)$ and let each arc $A_i \in A$ intersects at least two arcs in $A - \{A_i\}$, $i = 1, 2, ..., n$. Then $\gamma_{cnd}(G) = n - 1$ if and only if each pair of intersecting arcs $A_i, A_j$ $(i \neq j)$, each arc in $A - \{A_i, A_j\}$ intersects at least one of $A_i$ and $A_j$.

Proof: Let $A = \{A_1, A_2, ..., A_n\}$, $(n \geq 4)$ be a circular-arc family corresponding to a circular-arc graph $G$. Let $v_1, v_2, ..., v_n$ be the vertices in $G$ corresponding to arcs $A_1, A_2, ..., A_n$ respectively.

Assume each arc $A_i \in A$ intersects at most one arc in $A - \{A_i\}$, $i = 1, 2, ..., n$ and $\gamma_{cnd}(G) = n - 1$. Let $A_i$ and $A_j$ be two intersecting arcs in $A$. Let $D$ be a subset of $G$ such that $D$ is not a nil dominating set of $G$. Hence $\gamma_{cnd}(G) \leq n - 2$, which is a contradiction.

Hence each arc in $A - \{A_1, A_2\}$ intersects at least one of $A_1$ and $A_2$.

Conversely, assume each arc $A_i \in A$ intersects at least one of $A_1$ and $A_2$. Since $G \notin K_n$, $(n \geq 4)$, there exists at least two nonintersecting arcs in $A$. Hence $\gamma_{cnd}(G) = n - 1$.

Let $D$ be a ctn of $G$ with $|D| = n - 2$. Let $A_1, A_2$ be the arcs corresponding to the vertices $u, v$ in $V(G) - D$.

Then the arcs in $A$ corresponding to the vertices of $G$ in $V(G) - D$ are intersecting arcs and by assumption, $A_i$ corresponding to vertices in $D$ intersect at least one of $A_1$ and $A_2$. Then $D$ will not be a nil dominating set of $G$. Therefore, $|D| \geq n - 1$. Hence, $\gamma_{cnd}(G) = n - 1$.

Example 3.2: The circular-arc family $A = \{A_1, A_2, A_3, A_4\}$ corresponding to a circular graph $G$ for which $\gamma_{ctn}(G) = n = 1$ is given in Fig (3.2).

The circular-arc family satisfies all the conditions mentioned in Theorem 3.2, for $n = 5$. The complementary tree nil domination number $\gamma_{cnd}(G)$ of this graph is 4. The set of arcs due to minimum ctn of $G$ are $\{A_1, A_2, A_3, A_4\}$ and $\{A_1, A_2, A_3, A_4\}$.

Fig (3.2)

Remark 3.1: Let any two arcs in $A$ be intersecting arcs. If $A$ can be partitioned into two sets $X$ and $Y$ such that $|X| = m (1 \leq m < n - 1)$ and any pair of arcs in $X$ is nonintersecting and each arc in $X$ intersects exactly one arc in $Y$, then $\gamma_{cnd}(G) \geq m + 1$. 

Available Online at www.ijeecse.com
In the following, a necessary and sufficient condition that $\square_{ctnd}(G) = n - 1$, (where $\delta(G) = 1$) for a circular-arc graph $G$ is obtained.

**Theorem 3.3:** Let $A = \{A_1, A_2, \ldots A_n\}$, $(n \geq 3)$ be a circular-arc family analogous to a circular-arc graph $G$. For any two intersecting arcs $A_i, A_j$ $(i \neq j)$, each arc in $A - \{A_i, A_j\}$ intersects atmost one of $A_i$ and $A_j$. Then $\square_{ctnd}(G) = n - 1$ if and only if there exists a pair of intersecting arcs, say $A_l$ and $A_m$ $(l \neq m)$ in $A$ such that either (i) All the arcs in $A - \{A_l, A_m\}$ intersects $A_l$ or all the arcs in $A - \{A_l, A_m\}$ intersects $A_m$, $(1 \leq l, m \leq n, l \neq m)$ or (ii) Some of the arcs in $A - \{A_l, A_m\}$ intersects $A_l$ and the remaining arcs in $A - \{A_l, A_m\}$ intersects $A_m$, respectively.

Let $A_l$ and $A_m$ $(l \neq m)$ be any two intersecting arcs in $A$. If (i) or (ii) is satisfied, then either $V(G) - \{v_l\}$ or $V(G) - \{v_m\}$ is a $ctnd$ - set of $G$. Therefore, $\square_{ctnd}(G) \leq n - 1$. Also the vertices of $G$ corresponding to the arcs in $A - \{A_l, A_m\}$ are pendant vertices in $G$. Therefore by Remark 3.1., $\square_{ctnd}(G) \geq n - 2 + 1 = n - 1$ and hence $\square_{ctnd}(G) = n - 1$.

Conversely, assume $\square_{ctnd}(G) = n - 1$ and $A_l, A_m$ are any two intersecting arcs in $A$. If there exists an arc in $A - \{A_l, A_m\}$ intersecting none of $A_l$ and $A_m$, then $V(G) - \{v_l, v_m\}$ is a $ctnd$ - set of $G$ and hence $\square_{ctnd}(G) \leq n - 2$. Therefore

either (i) all the arcs in $A - \{A_l, A_m\}$ intersects $A_l$ or all the arcs in $A - \{A_l, A_m\}$ intersects $A_m$, $(1 \leq l, m \leq n, l \neq m)$ or (ii) some of the arcs in $A - \{A_l, A_m\}$ intersects $A_l$ and the remaining arcs in $A - \{A_l, A_m\}$ intersects $A_m$.

**Example 3.3:** The circular-arc family $A = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ corresponding to a circular graph $G$, for which $\square_{ctnd}(G) = n - 1$ is given in Fig (3.3).

The circular-arc family satisfies all the conditions mentioned in Theorem 3.3., for $n = 7$. The complementary tree nil domination number of this graph is 6. The set of arcs due to minimum $ctnd$ - sets of $G$ are $\{A_2, A_3, A_4, A_5, A_6\}$ and $\{A_1, A_2, A_3, A_4, A_5\}$.

In the following, In the following, a necessary and sufficient condition that $\square_{ctnd}(G) = 2$ for a circular-arc graph $G$ is obtained.

**Theorem 3.4:** Let $A = \{A_1, A_2, \ldots A_n\}$, $(n \geq 3)$ be a circular-arc family corresponding to a circular-arc graph $G$ and let $A_l$ and $A_j$ $(i \neq j)$ be any two intersecting arcs. Then $\square_{ctnd}(G) = 2$ if and only if all the arcs in $B = A - \{A_l, A_j\}$ intersect exactly one of $A_l$ and $A_j$, say $A_l$ and for any two intersecting arcs $A_s, A_r$ $(s \neq m)$ in $B$, each arc in $B - \{A_s, A_r\}$ intersects atmost one of $A_l$ and $A_m$.

**Proof:** Let $A = \{A_1, A_2, \ldots A_n\}$, $(n \geq 3)$ be a circular-arc family corresponding to a circular-arc graph $G$. Let $v_1, v_2, \ldots v_n$ be the vertices corresponding to the arcs $A_1, A_2, \ldots A_n$ respectively. Assume $\square_{ctnd}(G) = 2$. Let $D = \{v_i, v_j\}$ be a $\square_{ctnd}(G)$ - set of $G$. Since $V(G) - D$ is not a dominating set of $G$, there is a vertex, say $v \in D$ with $N(v) \subseteq D$. That is, $v_i$ is adjacent to $v$ in $G$. Also each vertex in $V(G) - D$ is adjacent to $v$. Therefore $A_l$ and $A_j$ are intersecting arcs in $A$ and all the arcs in $A - \{A_l, A_j\}$ intersect $A_l$. Since $V(G) - D$ is a tree, for any two intersecting arcs $A_s, A_r$, $(s \neq r)$ in $B$, each arc in $B - \{A_s, A_r\}$ intersects atmost one of $A_l$ and $A_m$.

Conversely, assume $A_l$ and $A_j$ $(i \neq j)$ are two intersecting arcs, and all the arcs in $B - \{A_l, A_j\}$ intersect $A_l$ and for any two intersecting arcs $A_s, A_r$ $(s \neq r)$ in $B$, each arc in $B - \{A_s, A_r\}$ intersects atmost one of $A_l$ and $A_m$. But there is no circular-arc graph $G$ for which $\square_{ctnd}(G) = 1$. Therefore $\square_{ctnd}(G) = 2$.

**Example 3.4:** The circular-arc family $A = \{A_1, A_2, A_3, A_4, A_5\}$ corresponding to a circular graph $G$, for which $\square_{ctnd}(G) = 2$ is given in Fig (3.4).

The circular arc family satisfies all the conditions mentioned in Theorem 3.4., for $n = 6$. The
complementary tree nil domination number of this graph is 2. The set of arcs due to minimum ctnd - sets of G is $\{A_1, A_2\}$.

IV. REFERENCES


