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Boundary Graph of a Graph

M. Bhanumathi[1], M. Kavitha[2]
[1]Associate Professor, Govt. Arts College for Women, Pudukkottai, Tamilnadu, India
[2]Research Scholar, Govt. Arts College for Women, Pudukkottai, Tamil nadu, India
[1]bhanu_ksp@yahoo.com, [2]kavisdev@gmail.com

***Abstract:** Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). The boundary graph of a graph G, denoted by $G_b(G)$ has the same set of vertices as G with two vertices u and v being adjacent in $G_b(G)$ if and only if either v is a boundary vertex of u in G or u is a boundary vertex of v in G. In this paper, we have defined boundary graph of a graph and found out boundary graph of some families of graphs. We have also studied some structural properties of boundary graph of a graph.*

***Keywords:** boundary vertex, boundary graph of a graph 2010 Mathematics Subject Classification: 05C12, 05C69.*

I. INTRODUCTION

Graphs discussed in this paper are undirected and simple. For graph theoretic terminology refer to Harary [4], Buckley and Harary [1]. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. A graph with p vertices and q edges is called a (p, q) graph.

The length of any shortest path between any two vertices u and v of a connected graph G is called the distance between u and v and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, The eccentricity e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$.The radius rad(G) is the minimum eccentricity of the vertices, whereas the diameter = diam(G) is the maximum eccentricity. If these two are equal in a graph, that graph is called self-centered graph with radius r and is called an r self-centered graph. For any connected graph G, rad(G) ≤ diam(G) ≤ 2rad(G). v is a central vertex if e(v) = r(G). The center C(G) is the set of all central vertices. For a vertex v, each vertex at a distance e(v) from v is an eccentric vertex of v.

The girth g(G) of the graph G, is the length of the shortest cycle(if any) in G.

A subgraph of G is a graph having all of its vertices and edges in G. It is a spanning subgraph if it contains all the vertices of G. If H is a subgraph of G, then G is a super graph of H. For any set S of vertices in G, the induced subgraph < S > is the maximal subgraph with vertex set S.

The complement $\bar{G}$ of a graph G is the graph with vertex set V(G) such that two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. A self-complementary graph is a graph, which is isomorphic to its complement.

A graph G is connected if there is a path joining each pair of vertices. A component of a graph is a maximal connected subgraph. If a graph has only one component, then it is connected. Otherwise it is disconnected. The diameter diam(G) of a connected graph G is the length of any, longest geodesic (diametral path). A clique of a graph is a maximal complete subgraph.

The concept of domination in graphs was introduced by Ore[7]. The concepts of domination in graphs originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. For details on γ(G), refer to [2].

A set D ⊆ V is said to be a dominating set in G, if every vertex in V–D is adjacent to some vertex in D. The cardinality of minimum dominating set is called the domination number and is denoted by γ(G). A dominating set D is an independent dominating set, if no two vertices in D are adjacent that is D is an independent set. The independent domination number is denoted by $\gamma_i(G)$. A dominating set D is a connected dominating set, if < D > is a connected subgraph of G. A dominating set D is a total dominating set, if < D > has no isolated vertices. The connected and total domination number are denoted by $\gamma_c(G)$ and $\gamma_t(G)$ respectively [2].

The concept of distance in graph plays a dominant role in the study of structural properties of graphs in various angles using related concept of eccentricity of vertices in graphs.

The set E_i denotes the set of vertices in G of eccentricity i [5].

A vertex v is a boundary vertex of u if d(u, w) ≤ d(u, v) for all w ∈ N(v). A vertex u can have more than one boundary vertex at different distance levels.

We know, G has at least two eccentric vertices. Also, all the eccentric vertices of G are boundary vertices of G. Hence, G has at least two boundary vertices.

A vertex v is called a boundary neighbour of u if v is a nearest boundary of u. The number of boundary neighbours of u is called the boundary degree of u. [3]

In 1985, [6] Jin Akiyama, Kiyoshi Ando, David Avis has defined Eccentric graphs of graphs.

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For any graph G, the eccentric graph G_e has the same set of vertices and any two vertices of G_e are adjacent if and only if one of the two vertices has maximum possible distance from the other. That is $V(G_e) = V(G)$ and $uv \in E(G_e) \Leftrightarrow d(u, v) = \min\{e(u), e(v)\}$, where e(u) and e(v) are eccentricities of u and v respectively in G.

Motivated by this, here we have defined boundary graph of a graph and studied its properties

In this paper, we define boundary graph $G_b(G)$ of a graph G. We find $G_b(G)$ for some families of graphs, and we study some properties of $G_b(G)$.

II. Boundary Graph

We define a boundary graph of a graph as follows.

A vertex v is a boundary vertex of u if $d(u, w) \leq d(u, v)$ for all $w \in N(v)$. A vertex v is called a boundary neighbour of u if v is a nearest boundary of u.

The *boundary graph* of a graph G, denoted by $G_b(G)$ has the same set of vertices as G with two vertices u and v being adjacent in $G_b(G)$ if and only if either v is a boundary vertex of u in G or u is a boundary vertex of v in G.

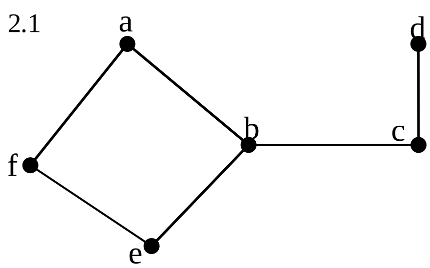
We define a boundary digraph as follows.

The *boundary digraph* of a graph G, denoted by BD(G) has the same set of vertices as G, and there is an arc from u to v in BD(G) if and only if v is a boundary vertex of u in G. Clearly, $G_b(G)$ is the underlying graph of BD(G).

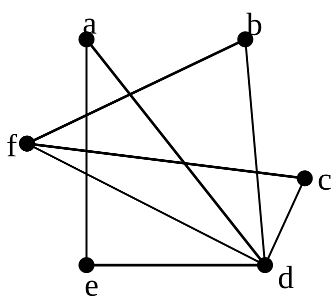
It is to be noted that given a graph G, the boundary graph $G_b(G)$ is uniquely defined, but given $G_b(G)$, there exist more than one graph H with $G_b(H) = G_b(G)$.

Example: 2.1

G:



$G_b(G)$



Observation: 2.1

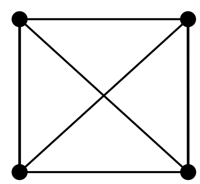
(i) Boundary graph has no isolated vertices since every vertex in a graph has at least one boundary vertex.

(ii) If G is a disconnected graph then $G_b(G)$ is also a disconnected graph but converse need not be true.

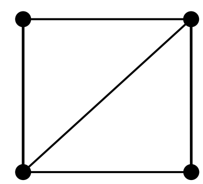
(iii) Two isomorphic graphs have their boundary graphs isomorphic but the converse need not be true always.

(iv) Eccentric graph of a graph G is always a subgraph of the boundary graph $G_b(G)$.

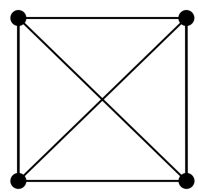
Example: 2.2



G



H



$G_b(H) = G_b(G)$

Theorem: 2.1

(i) $G_b(K_n) = K_n$.

(ii) $G_b(K_{1,n}) = K_{n+1}$.

(iii) $G_b(K_{m,n}) = K_m \cup K_n$, $m, n \geq 2$.

(iv) $G_b(C_{2n}) = nK_2$.

(v) $G_b(C_{2n+1}) = C_{2n+1}$.

(vi) $G_b(P_n) = K_2 + (n-2)K_1$.

(vii) $G_b(W_n) = K_1 + (K_n - C_n)$, $n \geq 4$, $G_b(W_3) = K_4$.

Proof:

(i) Assume $G = K_n$.

Every vertex of K_n is a boundary vertex. Hence every pair of vertices of G are adjacent in $G_b(K_n) = K_n$.

(ii) Let $G = K_{1,n}$.

Let u be the central vertex of G. All the pendant vertices are boundary vertices of u. Thus u is adjacent to all other vertices of G. The pendant vertices are boundary vertices to each other. Therefore, $G_b(K_{1,\ n})$ become a complete graph.

(iii) Let $G = K_{m,\ n}$, m, n ≥ 2.

Consider $V(G) = V_1 \cup V_2$. $|V_1| = m$ and $|V_2| = n$. Every vertex $u \in V_1(G)$ is a boundary vertex of other vertices of $V_1(G)$. In $G_b(K_{m,\ n})$, m vertices form a complete graph and any vertex of $V_1(G)$ is not adjacent to any other vertex of $V_2(G)$ in $G_b(K_{m,\ n})$. Similarly every vertex $v \in V_2(G)$ is a boundary vertex of other vertices of $V_2(G)$. Thus it follows that we get two complete components. Hence $G_b(K_{m,\ n}) = K_m \cup K_n$.

(iv) Let $G = C_{2n}$.

Let $v_1, v_2, \ldots, v_n, \ldots, v_{2n}$ be the vertices of C_{2n}. Every vertex of C_{2n} has exactly one boundary vertex. Thus in $G_b(C_{2n})$, every vertex of is adjacent to only one vertex. Therefore, $G_b(C_{2n}) = nK_2$.

(v) Let $G = C_{2n+1}$.

Let $v_1, v_2, \ldots v_{i-1}, v_i, v_{i+1}, \ldots, v_n, \ldots, v_{2n}, v_{2n+1}$ be the vertices of C_{2n+1}. Every vertex of C_{2n+1} has exactly two boundary vertices. Thus, every vertex is adjacent to two vertices in $G_b(G)$. Hence degree of each vertex in $G_b(C_{2n+1})$ is two. In $G_b(C_{2n+1})$, v_i is adjacent to v_{i+r} and v_{i+r+1}, where r is the radius of G. Hence $G_b(C_{2n+1}) = C_{2n+1}$.

(vi) Let $G = P_n$.

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of P_n. The pendant vertices v_1 and v_n are the boundary vertices of other vertices. Thus, v_1 and v_n are adjacent to other vertices of P_n and v_1 and v_n are adjacent in $G_b(P_n)$ since v_1 is a boundary vertex of v_n and vice versa. Hence we get a graph $G_b(P_n) = K_2 + (n-2)K_1$.

(vii) Let $G = W_n$, n ≥ 4.

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of C_n in W_n and v is a central vertex. For each vertex v_i all other vertices are the boundary vertices except v_{i-1}, v_{i+1} and v. For v, the remaining vertices $v_1, v_2, v_3, \ldots, v_n$ are the boundary vertices. Hence v is adjacent to all other vertices in $G_b(W_n)$ and every vertex v_i is adjacent to $(n-2)$ vertices in $G_b(W_n)$. It follows that $G_b(W_n)$ is $K_1 + (K_n - C_n)$.

Theorem: 2.2

If G has a pendant vertex then degree of that vertex in $G_b(G)$ is p–1.

Proof:

Suppose G has a pendant vertex u. u is the boundary vertex of all other vertices in G. In $G_b(G)$, u is adjacent to all other vertices. Hence the degree of u is p–1 in $G_b(G)$.

Cor 2.1: $G_b(G)$ is of radius one if G has a pendant vertex.

Theorem: 2.3

If G is not a star and has a pendant vertex then $G_b(G)$ is a bi-eccentric graph with diameter 2.

Proof:

Suppose G has a pendant vertex u. By the above corollary, the radius of $G_b(G)$ is one. Thus the eccentricity of other vertices is two. Hence $G_b(G)$ is a bi-eccentric graph with diameter 2.

Theorem: 2.4

If r(G) = 1 then $r(G_b(G)) = 1$.

Proof:

Let G be a graph with r(G) = 1. Consider a central vertex u. In $G_b(G)$, u is adjacent to all other vertices. Hence $r(G_b(G)) = 1$.

Theorem: 2.5

Let G be a graph without pendant vertices and has no triangles then $G_b(G + K_n) = \overline{G} + K_n$, n ≥ 1.

Proof:

Consider a graph G which has no triangle and has no pendant vertex. Let $H = G + K_n$.We have, $V(H) = V(G) \cup V(K_n)$. A vertex $u \in K_n$ is adjacent to all the vertices of a graph G in H.. Then all the vertices of H are the boundary vertices of u. Thus, in $G_b(H)$, a vertex $v \in V(K_n)$ is adjacent to all other vertices of $G_b(H)$. A vertex $v \in V(G)$ has a boundary vertex at distance at least 2 in G since G has no triangle and has no pendant vertices. So in $G_b(H)$, $v \in V(G)$ is not adjacent to any element of V(G). Thus, edges of $G_b(H)$ are the edges of $\overline{G}$ and edges from vertices of V(G) to vertices of $V(K_n)$.

Therefore, $G_b(G + K_n) = \overline{G} + K_n$.

The following theorems characterize graphs for which $G_b(G) = K_p$, $\overline{G}$ or G.

Theorem: 2.6

For any connected graph G of order p, $G_b(G) = K_p$ if and only if r(G) = 1 and no two vertices of eccentricity two are adjacent.

Proof:

Suppose $G_b(G) = K_p$ and assume $r(G) \neq 1$ and two vertices of eccentricity two are adjacent. Since G is connected there exists vertices u, v of G such that e(u) > 1, e(v) > 1 and $uv \in E(G)$ (ie) $u \in N(v)$. But this implies that u is not a farthest vertex of v and v is not a farthest vertex of u. Thus edge uv is not contained in G_b, which is a contradiction.

On the other hand, suppose r(G) = 1 and no two vertices of eccentricity two are adjacent. Let v be any vertex of G. Then if e(v) = 1, all other vertices are farthest vertices of v, hence all the edges from vertex v are present in G_b.

If $e(v) \neq 1$, e(v) = 2. Let w be any vertex in N(v). By the given condition e(w) = 1, so v is a farthest vertex of w and edge vw is in G_b. If $w \notin N(v)$, then d(v, w) = 2. In this case, w is a farthest vertex of v so edge vw is present in $G_b(G)$ but $vw \notin E(G)$. Therefore, for any two vertices u, v of G, $uv \in E(G_b)$. Thus $G_b(G) = K_p$.

Theorem: 2.7

Let G be a 2-self centered graph. $G_b(G) = \overline{G}$ if and only if for $v \in V(G)$, every $u \in N_1(v)$ has at least one successor in $N_2(v)$.

Proof:

Let G be a 2-self centered graph. Assume $G_b(G) = \overline{G}$. This implies that for a vertex $u \in V(G)$, every non adjacent vertex v is a boundary vertex of u. Therefore, for $v \in V(G)$, every $u \in N_1(v)$ has at least one successor in $N_2(v)$. (If there exist $u \in N_1(v)$ such that it has no successor in $N_2(v)$, then u is a boundary vertex of v. Thus, edge $uv \in E(G)$ is in $G_b(G)$. Hence, $G_b(G) \neq \overline{G}$, which is a contradiction.)

Conversely assume for $v \in V(G)$, every $u \in N_1(v)$ has at least one successor in $N_2(v)$. Every vertex in $N_2(v)$ is a boundary vertex of v and no vertex in $N_1(v)$ is a boundary vertex of v. Hence any two non adjacent vertices in G are adjacent in $G_b(G)$. Therefore, $G_b(G) = \overline{G}$.

Theorem: 2.8

Let G be a self centered graph with radius 2. $G_b(G) \cong G$ if and only if G is self complementary.

Proof:

Since G is self centered graph with radius 2, for $u \in V(G)$, each non-adjacent vertex of u is also a boundary vertex of u. Hence, $G_b(G)$ contains all the edges of $\overline{G}$. So $G_b(G) = G$ implies $G \cong \overline{G}$. This proves the theorem.

Theorem: 2.9

If r(G) = 1, then $G_b(G) = G$ if and only if $\langle V - E_1 \rangle_G$ is self complementary.

Proof:

Suppose $v \in E_1$. Then v has eccentricity one in $G_b(G)$. Thus if $v, w \in V - E_1$ they are adjacent in G if and only if they are non adjacent in $G_b(G)$. Therefore, any isomorphism from $G_b(G)$ to G must map vertices in $V - E_1$ to vertices in $V - E_1$, hence this induced subgraph of G must be self complementary.

Note:

(i) Odd cycles is a class of graphs for which $G_b(G) \cong G$.

(ii) Complete graphs is another class of graphs for which $G_b(G) = G$.

Theorem: 2.10

Let G be a connected graph. Let $u \in V(G)$ such that d(u) = k in G and u lies on a clique K_{k+1} in G. Then deg u in $G_b(G)$ is p-1.

Proof:

Let G be a connected graph with clique K_{k+1}. Consider a vertex u which lies on a clique with d(u) = k in G. u is a boundary vertex of other vertices of G. Hence degree of the vertex u is p-1 in $G_b(G)$.

Cor: 2.2 If G has a clique K_{k+1} and d(u) = k and u lies on a clique K_{k+1} in G then

(i) $\gamma(G_b(G)) = \gamma_i(G_b(G)) = 1$

(ii) $\gamma_t(G_b(G)) = 2$.

Proof:

Let G be a graph with a clique K_{k+1}; let d(u) = k and u lies on the clique K_{k+1}.

(i) By the above theorem deg u in $G_b(G)$ is p–1. Thus, D = {u} is a dominating set of $G_b(G)$ and is also an independent dominating set of $G_b(G)$. Hence $\gamma(G_b(G)) = \gamma_i(G_b(G)) = 1$.

(ii) follows from (i).

Naxt theorem gives the exact structure of $G_b(G)$, when G is a tree.

Theorem: 2.11

If T is a tree with p vertices and s pendant vertices then $G_b(T) = K_s + (n - s) K_1$.

Proof:

Let T be a tree with s pendant vertices. Every pendant vertex of a tree T is a boundary vertex of other vertices. Thus all the vertices of T are adjacent to each pendant vertex. The pendant vertices are adjacent to each other since the pendant vertices are the boundary vertices of each other. Thus $G_b(T)$ has a clique with s vertices. The non pendant vertices of T are adjacent to every vertex of a clique in $G_b(T)$. Hence $G_b(T)$ is of the form $K_s + (n - s) K_1$.

Observation: 2.2

(i) Let u be a pendant vertex of a tree T. In $G_b(T)$, degfree of u is $p - 1$. Thus $\Delta(G_b(T)) = p - 1$.

(ii) $G_b(T)$ is a bi eccentric graph with diameter 2.

(iii) $\gamma(G_b(T)) = 1$.

Theorem: 2.12

If T is a tree, then girth of $G_b(T)$ is three.

Proof:

Let T be a tree with n vertices. A tree T has at least two pendant vertices u and v. Consider a vertex $w \in V(T)$. u and v are the boundary vertices of w and u is a boundary vertex of v and vice versa. In $G_b(T)$, u and v are adjacent and also adjacent with w. Thus {u, v, w} form a cycle in $G_b(T)$. Hence girth of $G_b(T)$ is three.

Theorem: 2.13

Radius of $G_b(G)$ is one if G is any one of the following graph:

(i) G is complete.

(ii) G has a pendent vertex.

(iii) If G has a vertex u with d(u) = k and u lies on a clique K_{k+1} in G.

(iv) G is a graph with r(G) = 1 and $< V - E_1>_G$ is self complementary or E_2 is an independent set.

Proof: Follows from theorems 2.1, 2.2, 2.9 & 2.10.

Problem: Characterize graphs G for which Radius of $G_b(G)$ is one.

III. CONCLUSION

We have found out the boundary graph of some standard families of graphs. We have discussed the structure **of** $G_b(G)$ whenever G is a tree, G is a graph of radius one, G has pendent vertices etc.

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