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K-Gamma Distribution: Cumulant Generating Function and their Relation with Moments and Central Moments

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Abstract: ***The main purpose of this paper are cumulant generating function of k-gamma distribution in terms of parameter k, to find the relation between cumulants and moments of k-gamma distribution, also calculate karl-pearson coefficient of beta and gamma of k-gamma distribution. In probability theory and statistics, the cumulants*** κ_n ***of a probability distribution are a set of quantities that provide an alternative to the moments of the distribution. The moments determine the cumulants in the sense that any two probability distributions whose moments are identical will have identical cumulants as well, and similarly the cumulants determine the moments.***

Keyword: Pochhammer k-symbol, k-gamma functions, Moment Generating function of k-gamma distribution, Cumulant generating function of k-gamma distribution, karl-pearson coefficient of beta and gamma of k-gamma distribution,etc.

I. INTRODUCTION

We give definitions related to gamma function, k-gamma function which is provided by researchers in [1]. Also we use some relation of statistics and probability [3]. Some writers prefer to define the cumulant-generating function as the natural logarithm of the characteristic function, which is sometimes also called the ***second*** **characteristic function.** The cumulant generating function if it exists, is infinitely differentiable and convex, and passes through the origin. Its first derivative ranges monotonically in the open interval from the infimum to the supremum of the support of the probability distribution, and its second derivative is strictly positive everywhere it is defined, except for the degenerate distribution of a single point mass. The cumulant-generating function exists if and only if the tails of the distribution are majorized by an exponential decay.

A. Pochhammer K Symbol

The Function defined

$$(\alpha)_{n,k} = \begin{Bmatrix} \alpha(\alpha+k)(\alpha+2k)\cdots(\alpha+(n-1)k) \; for\; n\geq 1, \alpha\neq 0 \\ 1 \qquad\qquad if\; n=0 \end{Bmatrix} \quad (1)$$

B. k- Gamma Function

The integral form of k-gamma function is given as

$$\Gamma_k(\mathrm{y}) = \int_0^\infty \mathrm{t}^{\mathrm{y}-1}\ \mathrm{e}^{-\frac{\mathrm{t}^\mathrm{k}}{\mathrm{k}}}\mathrm{dt} \quad (2)$$

Also the properties provided by researchers are as follows:

$$\Gamma_k(\mathrm{y}+\mathrm{k}) = \mathrm{y}\,\Gamma_k(\mathrm{y})$$

$$(y)_{n,k} = \frac{\Gamma_k(\mathrm{y}+\mathrm{k})}{\Gamma_k(\mathrm{y})}$$

$$\Gamma_k(\mathrm{k}) = 1 \qquad \mathrm{k}>0$$

$$\Gamma_k(\mathrm{y}) = \mathrm{k}^{\frac{\mathrm{y}}{\mathrm{k}}-1}\Gamma\frac{y}{k}$$

$$\Gamma_k(\mathrm{yk}) = \mathrm{k}^{y-1}\Gamma y$$

$$\Gamma_k(\mathrm{nk}) = \mathrm{k}^{\mathrm{n}-1}(\mathrm{n}-1)! \qquad \mathrm{k}>0, \mathrm{n}\epsilon\mathrm{N}$$

C. k- Gamma Distribution

Let Z be a continuous random variable of k-gamma distribution with parameter m > 0 and k > 0 if its probability density function is defined in [1] is:

$$f_k(z) = \begin{cases} \frac{1}{\Gamma_k(\mathrm{m})} z^{m-1} e^{\frac{-z^k}{k}} & , 0\leq z<\infty, k>0 \\ 0 & elsewhere \end{cases} \quad (3)$$

D. Moment Generating function of k-gamma distribution

The moment generating function of continuous random variable Z of k-gamma distribution in terms of parameter k > 0 is defined in [1] is:

$$M_{0,k}(t) = E_k\left(e^{tz^k}\right) = \int_0^\infty \frac{1}{\Gamma_k(\mathrm{m})} e^{tz^k} z^{m-1} e^{\frac{-z^k}{k}} dz = (1-kt)^{-\frac{m}{k}}, |kt|<1 \quad (4)$$

E. Moments of k-gamma distribution

The r[th] moment in terms of k is given by [1] is:

$$\mu'_{r,k} = m(m+k)(m+2k)(m+3k)\ldots..(m+(r-1)k) \quad (5)$$

When r=1 $\mu'_{1,k} = m,$ (6)

r=2 $\mu'_{2,k} = m(m+k)$ (7)

r=3 $\mu'_{3,k} = m(m+k)(m+2k)$ (8)

r=4 $\mu'_{4,k} = m(m+k)(m+2k)(m+3k)$ (9)

F. The relation between Moments and Central Moments

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The relation is defined in [3] is:

$$\mu_1 = 0 \quad (10)$$

$$\mu_2 = \mu_2' - (\mu_1')^2 \quad (11)$$

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3 \quad (12)$$

$$\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2'(\mu_{1k}')^2 - 3(\mu_1')^4 \quad (13)$$

G. Karl-pearson beta and gamma coefficient

The Karl-Pearson beta and gamma coefficient for central moment is defined in [3] is:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \quad (14)$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \quad (15)$$

$$\gamma_2 = \beta_2 - 3 \quad (16)$$

$$\gamma_1 = \sqrt{\beta_1} \quad (17)$$

II. Cumulant Generating Function of K-Gamma Distribution

Major headings In this section, we find the cumulant generating function of k-gamma function which we represent by $G_{0,k}(t)$. The cumulant generating function is the logarithm of moment generating function and defined as

$$G_{0,k}(t) = log M_{0,k}(t) \quad (18)$$

Using eq^n (4) in (18)

$$G_{0,k}(t) = \log(1-kt)^{-\frac{m}{k}}$$
$$= -\frac{m}{k} log(1-kt) \qquad |kt| < 1$$
$$= \frac{m}{k}\left[kt + \frac{(kt)^2}{2} + \frac{(kt)^3}{3} + \frac{(kt)^4}{4} + \cdots\right] \quad (19)$$

κ_1 = Coefficient of t in $G_{0,k}(t)$ (20)

κ_2 = Coefficient of $\frac{t^2}{2!}$ in $G_{0,k}(t)$ = mk (21)

κ_3 = Coefficient of $\frac{t^3}{3!}$ in $G_{0,k}(t)$ = 2m k^2 (22)

κ_4 = Coefficient of $\frac{t^4}{4!}$ in $K_{0,k}(t)$ = 6m k^3 (23)

Here $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ are first four cumulants of k-gamma distribution.

III. Relation Between Cumulants And Moments of K-Gamma Distribution

a) We have by equations (6) to (9) and (20) to (23).

$$\mu_{1,k}' = m = \kappa_1 \quad (24)$$

$$\mu_{2,k}' = m(m+k) = mk + m^2 = \kappa_2 + \kappa_1^2 \quad (25)$$

$$\mu_{3,k}' = m(m+k)(m+2k) = 2m\,k^2 + 3mk(m) + m^3{}_3$$
$$= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3 \quad (26)$$

$$\mu_{4,k}' = m(m+k)(m+2k)(m+3k)$$
$$= \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4 \quad (27)$$

Here, we can see "First moment is equal to first degree polynomial of cumulant of k-gamma distribution", "Second degree polynomial of cumulant is equal to econd moment of k-gamma distribution " and so on.

b) Let central moment (moment about mean) is defined by $\mu_{r,k}$ where r=1,2,3,4…

By the equations (10) to (13) and (24) to (27) we have:

$$\mu_{1,k} = 0 \quad (28)$$

Using the eq^n (30) and (31) of [1].

$$\left[\begin{array}{l} \mu_{1,k} = E_k(z-\bar{z}) = \int_0^\infty (z-\bar{z}) f_k(z) dz = \int_0^\infty \frac{1}{\Gamma_k(\mathrm{m})}(z-\bar{z}) z^{m-1} e^{\frac{-z^k}{k}} dz \\ = \frac{1}{\Gamma_k(\mathrm{m})}\int_0^\infty (z) z^{m-1} e^{\frac{-z^k}{k}} dz - \frac{(\bar{z})}{\Gamma_k(\mathrm{m})}\int_0^\infty z^{m-1} e^{\frac{-z^k}{k}} dz \\ \mu_{1,k} = (\bar{z}) - (\bar{z}) = 0 \end{array}\right]$$

$$\mu_{2,k} = \mu_{2,k}' - (\mu_{1,k}')^2 = m(m+k) - m^2 = mk = \kappa_2$$
(29)

$$\mu_{3,k} = \mu_{3,k}' - 3\mu_{2,k}' \mu_{1,k}' + 2(\mu_{1,k}')^3$$
$$= m(m+k)(m+2k) - 3m(m+k)m + 2m^3$$
$$\mu_{3,k} = 2m\,k^2 = \kappa_3 \quad (30)$$

$$\mu_{4,k} = \mu_{4,k}' - 4\mu_{3,k}' \mu_{1,k}' + 6\mu_{2,k}' (\mu_{1,k}')^2 - 3(\mu_{1,k}')^4$$
$$= 6m\,k^3 + 3m^2 k^2 = \kappa_4 + 3\kappa_2^2 \quad (31)$$

IV. Karl Pearson Beta And Gamma Coefficient of K-Gamma Distribution

In this section we find the karl pearson beta and gamma coefficient of k-gamma distribution using the relation (14) to (17) and (28) to (31) :

$$\beta_1 = \frac{\mu_{3,k}^2}{\mu_{2,k}^3} = \frac{(2m\,k^2)^2}{(mk)^3} = \frac{4k}{m} \quad (32)$$

$$\beta_2 = \frac{\mu_{4,k}}{\mu_{2,k}^2} = \frac{6m\,k^3 + 3m^2k^2}{m^2k^2} = \frac{3(2k+m)}{m} \quad (33)$$

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{4k}{m}} \quad (34)$$

$$\gamma_2 = \beta_2 - 3 = \frac{3(2k+m)}{m} - 3 = \frac{6k}{m} \quad (35)$$

V. Conclusion

In this paper we are genrating a cumulant generating function of k-gamma distribution as$\kappa_1, \kappa_2, \kappa_3, \kappa_4$.we also find the relation that "the moment of k-gamma distribution are polynomial function of cumulants of k-gamma distribution". We calculate beta and gamma coefficient of k-gamma distribution for any k >0. The first cumulant is shift-equivariant; all of the others are shift-invariant. This means that, if we denote by $\kappa_n(X)$ the *n*th cumulant of the probability distribution of the random variable *X*, then for any constant. The moments determine the cumulants in the sense that any two probability distributions whose moments are identical will have identical cumulants as well, and similarly the cumulants determine the moments. In some cases theoretical treatments of problems in terms of cumulants are simpler than those using moments.

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